Vol.4 Issue 1, Pg.: 38-55
January 2016

INTERNATIONAL JOURNAL OF RESEARCH IN COMPUTER APPLICATIONS AND ROBOTICS
ISSN 2320-7345

IMAGE ANALYSIS USING A NEW DEFINITION OF MATHEMATICAL MORPHOLOGY FOR BINARY IMAGE

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Abstract: - What the algebra of convolution does for linear systems, the algebra of mathematical morphology does for shape. Since shape is a prime carrier of information in machine vision, there should be little surprise about the importance of the mathematical morphology. Now adays the operations of mathematical morphology are very much useful in object or defect identification required in industrial vision applications and target detection useful to military applications. A new definition for binary morphology covering the operation of dilation, erosion, opening and closing and their relations are presented in this paper. Examples are presented for each morphological operation.

Keywords: Morphology, dilation, erosion, opening, closing, shape analysis.

1. Introduction

Mathematical morphology has evolved as a useful tool for various image-processing tasks (Giardina & Daugherty, 1988), (Serra, 1982). An algebraic system of operators, such as those of mathematical morphology, is useful to the processing of digital images that are based on shape. "Morphology" can literally be taken to mean, "doing things to shapes". "Mathematical morphology" then, by extension, means using mathematical principals to do things to shapes. It treats an image as an ensemble of sets rather than signal. Its language is that of the set theory and operations are defined in terms of the iterations between the object and the structuring elements. The operators are able to decompose a complex shape structure into its meaningful parts and separate the meaningful parts from its extraneous parts. Morphological operators and their compositions are able to identify the underline shapes and reconstruct the best possible from their distorted noisy forms. Morphological operations can simplify image data, preserving their essential shape characteristics and eliminates irrelevancies. As the identification and decomposition of objects, object features, object surface defects and assembly defects correlate directly with shape; it is only natural that mathematical morphology has an essential structural role to play in machine vision (Haralick *ei al*., 1987), (Serra, 1986). Majman and Tolbot (2010) have presented a survey of the state of the art in mathematicl morphology.

Now a day mathematical morphological tools are one of the most effective tool in computer vision and morphological image processing is currently an emerging research topic (Tankyerych *et al*., 2013), (Miranda and Mansilla, 2014), (Ronse, 2014). Nakagawa and Rosenfeld (1978) first discussed the use of neighbourhood minimum and maximum operators for shrinking and expanding operations on two-valued digital pictures, which are useful for noise removal, as well as for detecting dense regions and elongated parts of objects. Sternberg (1982) has extended the work in generalised form. Peleg and Rosenfeld (1981) use greyscale morphology to generalise the medial axis transform to greyscale imaging. Tolbot and Appleton (2007) have presented a new ordered implementations of the complete and incomplete path opening and closing operators. Peleg *et al*. (1984) use greyscale morphology to measure changes in texture properties as a function of resolution. Werman and Peleg (1985) use greyscale morphology for texture feature extraction. Favre *at al*. (1985) use greyscale morphology for the detection of platelet thrombosis in cross sections of blood vessels. Coleman and Sampson (1985) use greyscale morphology on range data imagery to help mate a robot gripper to an object. Haralick *et al.* (1987) have defined greyscale morphological operations of dilation and erosion. First they introduced the concept of the top surface of a set and the related concept of the umbra of a surface. Then greyscale dilation was defined as the surface of the dilation of the umbrae. For this definition, they have proposed that the greyscale dilation can be computed in terms of a maximum operation and a set of addition operations. Similarly, the grayscale erosion can be evaluated in terms of a minimum operation and a set of subtraction operations. Ghosh (1996) use mathematical morphological operations of boundary represented geometric objects. Li *et al*. (1997) discussed a set of texture features which is based on morphological residues of opening and closing by reconstruction. Cuisenaire (2004) shows how common binary mathematical morphology operators can be adapted so that the size of the structuring element can vary across the image pixels. He shows that when the structuring elements are balls of a metric, locally adaptable erosion and dilation can be efficiently implemented as a variant of distance transformation algorithms.

Many scientists used mathematical morphological operations for feature enhancement (Chaudhuri *et al*., 2012), removing noise (Ze-Feng *et al*., 2007) and edge detection (Zhao *et al*., 2006). In mathematical morphology there are two basic functions working oppositely – erosion and dilation or those compositions – opening and closing. Chaudhuri *et al.* (2012) have suggested alternate sequential filtering, which is a directional morphological filtering for enhancing the road like structures of the image. Chaudhuri *et al.* (2012) have proposed the enhancement technique that is a combination of opening and closing operations using the same structuring element along the homogeneous direction with respect to the image and the road template. Morphological filters base on a simple idea of alternating two operations, opening and closing – removing noise, edge detection from the image (Kupidura and Koza, 2008). Kowalczyk *et al*. (2008) have suggested the effectiveness of morphological operations, which are used for the preparing of the close-range images for automatic correlations, especially the effectiveness of the edge preserving of the different types of filteration.

A new definition for binary morphology covering the operation of dilation, erosion, opening and closing and their relations are presented in this paper. The basic two operations of mathematical morphology are dilation and erosion. The proposed definitions for dilation and erosion are max-min and min-min operations, respectively. Several set theoretic properties are also discussed in this paper. Basic definitions of morphological operations like dilation, erosion, opening and closing for binary image are reported in Section 2. Section 3 presents the new definition of dilation and erosion operations for binary image. Several set theoretic properties are also discussed in Section 3. Opening and closing and their several set theoretic properties for binary image are discussed in Section 4.

2. Basic definitions of morphological operations

Mathematical morphology treats an image as an ensemble of sets rather than signal. Its language is that of the set theory and operations are defined in terms of the interaction between the object and the structuring element.

2.1 Binary dilation

Dilation is the morphological transformation that combines two sets using vector addition of set elements. Minkowski first used binary dilation and in the mathematics literature is called *Minkowski addition.* If *A* and *B* are two sets in *N*-space (E^N) with elements $a=(a_1,a_2,\ldots,a_N)$ and $b=(b_1,b_2,\ldots,b_N)$. The *dilation* of *A* and *B* is denoted by $A \oplus B$ and is defined by

$$A \oplus B = \{c \in E^N \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

2.2 Binary Erosion

Erosion is the mathematical dual of dilation. It is the morphological transformation that combines two sets using as its basis set containment. If *A* and *B* are two sets in *N*-space (E^N) with elements $a=(a_1,a_2,\ldots,a_N)$ and $b=(b_1,b_2,\ldots,b_N)$. The *erosion* of *A* and *B* is denoted by $A\Theta B$ and is defined by

$$A\Theta B = \{x \in E^N \mid x + b \in A \text{ for every } b \in B\}$$

2.3 Opening and closing

Dilation and erosion operations are very much useful for shape recognition. Dilation and erosion are usually operated in pairs, either dilation of an image followed by the erosion of dilated result, or an erosion of the image

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followed by the dilation of the eroded result. Opening and closing stand to morphology exactly as the orthogonal projection operator stands to linear algebra. An orthogonal projection operator is idempotent and selects that part of a vector that lies in a given subspace. Similarly, opening and closing provide the means by which given sub-shapes and super-shapes of a complex shape can be selected.

The *opening* of image B by structuring element K is denoted by $B\text{o}K$ and is defined by $B\text{o}K = (B \ominus K) \oplus K$.

The *closing* of image B by structuring element K is denoted by $B \bullet K$ and is defined by $B \bullet K = (B \oplus K) \ominus K$.

3. New definitions of dilation and erosion

The binary morphological operations dilation and erosion are presented in this section. The operators are max-min and min-min operators, respectively instead of Minkowski addition and subtraction operations.

3.1 Dilation

The proposed dilation operation has the same complexity as convolution. However, instead of doing the summation of products as in convolution, a maximum of minimums is performed. The binary image F is a function $f(x, y)$ where $(x, y) \in E^2$ is pixel position and $f(x, y)$ is the value either 1 or 0 at the pixel (x, y).

Definition 1: Let $I(r,c)$ and $H(i, j)$ be subsets of F. The dilation of $I(r,c)$ by $H(i, j)$ is denoted by $J_{I,H}(r,c)$ or $J_{I,H}$ and is defined by

$$J_{I,H} = J_{I,H}(r,c) = \max_{(r-i,c-j)\in I}\left[\min_{(i,j)\in H}\{I(r-i,c-j), H(i,j)\}\right]$$
$$= \{x \mid x = \max[\min(a,b)],\ \text{for some } a \in I(r-i,c-j) \text{ and } b \in H(i,j)\}$$

where $I(r-i,c-j)$ is the value at $(r-i,c-j)$ of the image set and $H(i,j)$ is the value at (i,j) of the structuring element set.

This operation of dilation is shown with an example as below. Origins are placed at the arrow positions of the image set and structuring element set and the pixels with a black dot (•) have a non-zero value 1. Since binary images have grey values 0 or 1 i.e., the foreground is 1 and background is 0. So the definition 1 satisfies the dilation of binary morphology. Figs. 1(a) and 1(b) shows the binary image set and the structuring element image set of sizes 64×64 and 9×9 respectively. The dilated image is shown in Fig. 1(c).

Example:

→↓	•	•	•		
	•	•			
	•	•	•		
•	•		•		
	•		•		

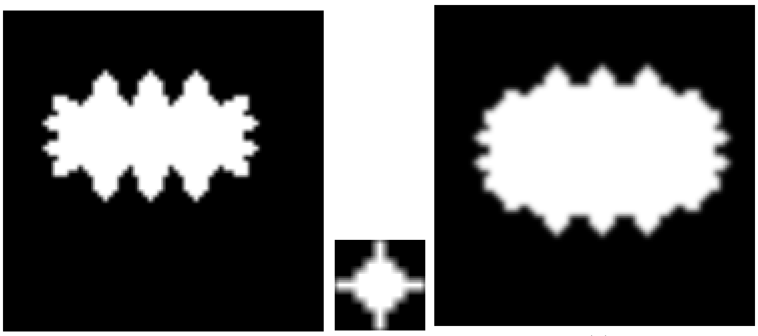
I

→↓ •	•

H

→↓	•	•	•	•	
	•	•	•		
	•	•	•	•	
•	•	•	•	•	
	•	•	•	•	

$J_{I,H}$



(a) (b) (c)

Fig. 1: Binary Image (a) Image of size 64×64, (b) structuring element of size 9×9, (c) Dilated image

It can be easily prove the following propositions for dilation operations.

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Proposition 1: Dilation operation is commutative i.e. $J_{I,H} = J_{H,J}$.

Proof : $J_{I,H} = \max\limits_{(r-i,c-j)\in I}\left[\min\limits_{(i,j)\in H}\{I(r-i,c-j),H(i,j)\}\right]$

$$= \{x \mid x = \max[\min(a,b)],\ \text{for some}\ a \in I(r-i,c-j)\ \text{and}\ b \in H(i,j)\}$$

$$= \{x \mid x = \max[\min(b,a)],\ \text{for some}\ b \in H(i,j)\ \text{and}\ a \in I(r-i,c-j)\}$$

Now replace *i* by *r-i* and *j* by *c-j* then

$$J_{I,H} = \{x \mid x = \max[\min(b,a)],\ \text{for some}\ b \in H(r-i,c-j)\ \text{and}\ a \in I(i,j)\}$$

$$= \max_{(r-i,c-j)\in H}\left[\min_{(i,j)\in I}\{H(r-i,c-j),I(i,j)\}\right] = J_{H,I}.$$

Some dilations by structuring elements larger than the neighbourhood size by iteratively dilating with a sequence of neighbourhood structuring elements can be implemented by neighbourhood connected image processors. The form $(A\oplus B)\oplus C$ takes less operations than $A\oplus(B\oplus C)$ because a brute force dilation by $B\oplus C$ might take as many as N^2 operations while dilating *A* by *B* and then dilating the result by *C* could take as few as $2N$ operations, where *N* is the number of elements in *B* and *C*. This computational complexity advantage is not as strong for machines which can implement dilation only as neighbourhood operations [3].

Proposition 2: Dilation operation is associative i.e. $J_{(I,H),S} = J_{I,(H,S)}$.

Proof : $J_{(I,H),S} = \max\limits_{(r-k,c-l)\in I}\left[\min\limits_{(k,l)\in S}\left\{\left(\max\limits_{(r-i,c-j)\in I}\left[\min\limits_{(i,j)\in H}\{I(r-k-i,c-l-j),H(i,j)\}\right]\right),S(k,l)\right\}\right]$

$$= \{x \mid x = \max[\min\{(\max[\min\{a,b\}]),d\}],\ \text{for some}\ a \in I(r-k-i,c-l-j), b \in H(i,j)$$
$$\text{and}\ d \in S(k,l)\}$$

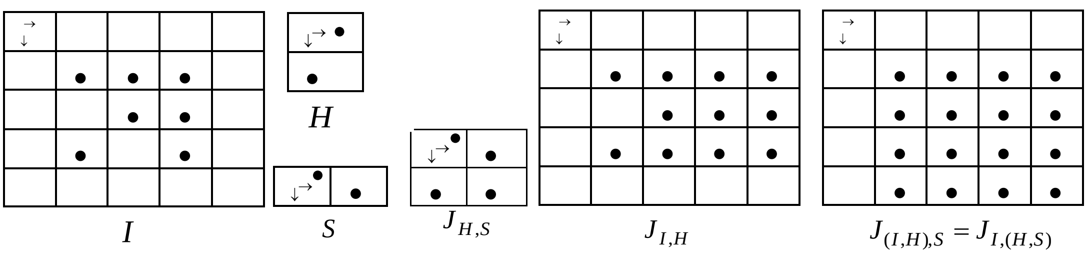
$$= \{x \mid x = \max[\min\{a,(\max[\min\{b,d\}])\}],\ \text{for some}\ a \in I(r-k-i,c-l-j),$$
$$b \in H(i,j)\ \text{and}\ d \in S(k,l)\}.$$

Now replace *i* by *i-k* and *j* by *j-l* then

$$J_{(I,H),S} = \{x \mid x = \max[\min\{a,(\max[\min\{b,d\}])\}],\ \text{for some}\ a \in I(r-k-i,c-l-j), b \in H(i,j)$$
$$\text{and}\ d \in S(k,l)\}$$

$$= \max_{(r-i,c-j)\in I}\left[\min_{(i,j)\in H}\left\{I(r-i,c-j),\left(\max_{(i-k,j-l)\in H}\left[\min_{(k,l)\in S}\{H(i-k,j-l),S(k,l)\}\right]\right)\right\}\right] = J_{I,(H,S)}.$$

Example :



Definition 2: Let $I_1(i,j)$ and $I_2(i,j)$ are two subsets of F. The union of $I_1(i,j)$ and $I_2(i,j)$ is denoted by $I_1 \cup I_2 \subseteq F$ and is defined by $I_1 \cup I_2 = \{x \mid x = \max(a,b),\ \text{for}\ a \in I_1(i,j)\ \text{and}\ b \in I_2(i,j)\}$.

Definition 3: Let $I(r,c)$ be a subset of F and $X = (i,j) \in E^2$. The translation of *I* by *X* is denoted by I_X and is defined by $I_X = I_X(r,c) = I(r+i,c+j) = \{a \mid a \in I(r,c)\}$.

Proposition 3: Let $I(r,c)$ and $H(i,j)$ be subsets of F then $J_{I,H} = \cup_{z\in H} I_z$, where (r,c) and $(i,j) \in E^2$.

Proof : Let $x(r_0,c_0) \in J_{I,H}$

$$\Rightarrow x(r_0,c_0) \in \max_{(r-i,c-j)\in I}\left[\min_{(i,j)\in H}\{I(r-i,c-j),H(i,j)\}\right]$$

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$\Rightarrow x(r_0, c_0) = \max[\min\{a, b\}]$, for some $a \in I(r-i, c-j)$ and $b \in H(i, j)$.

The value of the considering pixel (element) of structuring element set $H(i, j)$ is always 1 for binary image; so $a \le b$. Therefore, $x(r_0, c_0) = a \in I(r-i, c-j)$.

Replace -*i* by *i* and -*j* by *j*, we have $x(r_0, c_0) \in I(r+i, c+j) = I_z(r, c)$ where $z = (i, j)$.

Therefore $x(r_0, c_0) \in \cup_{z \in H} I_z$

So $J_{I,H} \subseteq \cup_{z \in H} I_z$ (1)

Similarly we can prove $\cup_{z \in H} I_z \subseteq J_{I,H}$ (2)

So from (1) and (2) we have $J_{I,H} = \cup_{z \in H} I_z$

It is important to know that the dilation of a shifted image by a structuring element set provides the same result if we first dilate the image by the same structuring element set and then it shifts.

Proposition 4: Dilation operation is translation invariance i.e. $J_{I_X,H} = (J_{I,H})_X$.

Proof : $y(r_0, c_0) \in J_{I_X,H}$ if and only if $y(r_0, c_0) = \max[\min\{z, b\}]$, for some $z \in I(r-i, c-j)_X$ and $b \in H(i, j)$. Let $X = (r_1, c_1)$.

Now $z \in I(r-i, c-j)_{X=(r_1,c_1)}$ implies $z \in I(r-i+r_1, c-j+c_1)$ i.e. $z \in I(r+r_1-i, c+c_1-j)$. So $y(r_0, c_0) \in J_{I_X,H}$ if and only if $y(r_0, c_0) = \max[\min\{z, b\}]$, for some $z \in I(r+r_1-i, c+c_1-j)$ and $b \in H(i, j)$.

i.e. $y(r_0, c_0) \in J_{I_X,H} \Leftrightarrow y(r_0, c_0) \in \max\limits_{(r+r_1-i,\, c+c_1-j) \in I} \left[\min\limits_{(i,j) \in H} \{I(r+r_1-i, c+c_1-j), H(i, j)\} \right]$

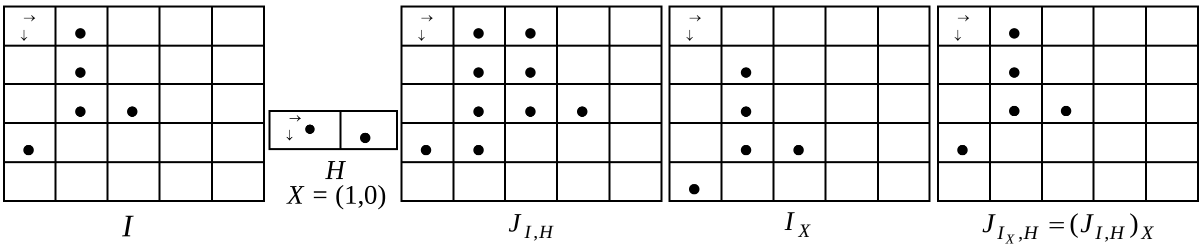
i.e. $\Leftrightarrow y(r_0, c_0) \in J_{I,H}(r+r_1, c+c_1)$

i.e. $\Leftrightarrow y(r_0, c_0) \in \left(J_{I,H}(r, c)\right)_{X=(r_1,c_1)}$

i.e. $\Leftrightarrow y(r_0, c_0) \in \left(\max\limits_{(r-i,c-j) \in I} \left[\min\limits_{(i,j) \in H} \{I(r-i, c-j), H(i, j)\} \right] \right)_X$

Therefore, $J_{I_X,H} = (J_{I,H})_X$

Example :



Proposition 4 can be extended in a straight way for *n* structuring element sets by chain rule.

Corollary 1: $J_{I,H_1,H_2,\ldots,(H_m)_X,\ldots,H_n} = \left(J_{I,H_1,H_2,\ldots,H_m,\ldots,H_n}\right)_X$.

If the image is shifted in one direction and the structuring element set is shifted in the opposite direction then the dilation of the shifted image and the shifted structuring element set gives the same result as that of dilation of the image and the structuring element set.

Corollary 2: $J_{I_X,H_{-X}} = J_{I,H}$.

Proof : We know $J_{I_X,H} = \left(J_{I,H}\right)_X$. So $J_{I_X,H_{-X}} = \left(J_{I,H_{-X}}\right)_X = \left(J_{I,H}\right)_{X-X} = J_{I,H}$.

Definition 4: $I_1(i, j)$ is said to be proper subset of $I_2(i, j)$ if every element of $I_1(i, j)$ is an element of $I_2(i, j)$. But there exists at least one element of $I_2(i, j)$, which does not belong to $I_1(i, j)$. This is denoted by $I_1(i, j) \subseteq I_2(i, j)$.

It is very interesting that the dilation preserves order that is, if of two images one is subset of other and are dilated by the same structuring element set then the dilated images also maintained the same order. Operators having this property are called *increasing operator*.

Proposition 5: Dilation is increasing i.e. if $I_1(r-i,c-j) \subseteq I_2(r-i,c-j)$ then $J_{I_1,H} \subseteq J_{I_2,H}$.

Proof : Let $x(r_0,c_0) \in \max\limits_{(r-i,c-j)\in I_1}\left[\min\limits_{(i,j)\in H}\{I_1(r-i,c-j),H(i,j)\}\right]$

Then for some $a \in I_1(r-i,c-j)$ and $b \in H(i,j)$, $x(r_0,c_0) = \max[\min\{a,b\}]$. Since $a \in I_1(r-i,c-j)$ and $I_1(r-i,c-j) \subseteq I_2(r-i,c-j)$ so $a \in I_2(r-i,c-j)$.

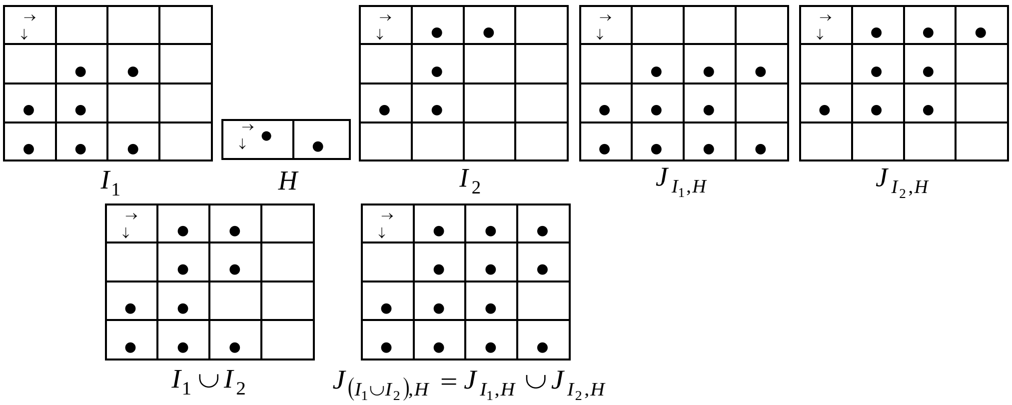
So by the definition of dilation $x(r_0,c_0) \in \max\limits_{(r-i,c-j)\in I_2}\left[\min\limits_{(i,j)\in H}\{I_2(r-i,c-j),H(i,j)\}\right]$

Therefore $J_{I_1,H} \subseteq J_{I_2,H}$.

Proposition 6: Dilating a union of two sets is also the union of dilation i.e. $J_{(I_1\cup I_2),H} = J_{I_1,H} \cup J_{I_2,H}$.

Proof : $x(r_0,c_0) \in \max\limits_{\substack{(r-i,c-j)\in I_1\\(r_1-i,c_1-j)\in I_2}}\left[\min\limits_{(i,j)\in H}\{I_1(r-i,c-j)\cup I_2(r_1-i,c_1-j),H(i,j)\}\right]$ if and only if for some $a \in I_1(r-i,c-j)\cup I_2(r_1-i,c_1-j)$ and $b \in H(i,j)$ such that $x(r_0,c_0) = \max[\min\{a,b\}]$. But $a \in I_1(r-i,c-j)\cup I_2(r_1-i,c_1-j)$ means $a = \max(e,d)$ for $e \in I_1(r-i,c-j)$ and $d \in I_2(r_1-i,c_1-j)$.
So $x(r_0,c_0) = \max[\min\{\max(e,d),b\}] = \max[\max\{\min(e,b)\},\max\{\min(d,b)\}]$.
Therefore if and only if
$x(r_0,c_0) \in \max\limits_{(r-i,c-j)\in I_1}\left[\min\limits_{(i,j)\in H}\{I_1(r-i,c-j),H(i,j)\}\right] \cup \max\limits_{(r^1-i,c^1-j)\in I_2}\left[\min\limits_{(i,j)H}\{I_2(r^1-i,c^1-j),H(i,j)\}\right]$
Hence $J_{(I_1\cup I_2),H} = J_{I_1,H} \cup J_{I_2,H}$

Example :



Proposition 7: $J_{I,(H\cup S)} = J_{I,H} \cup J_{I,S}$.

Proof : Since dilation is commutative so $J_{I,(H\cup S)} = J_{(H\cup S),I}$.

Now $J_{I,(H\cup S)} = J_{(H\cup S),I} = J_{H,I} \cup J_{S,I}$ (by proposition 6)

$= J_{I,H} \cup J_{I,S}$ (by commutative property).

Definition 6: The intersection of two sets $I_1(i,j)$ and $I_2(r,c)$ is denoted by $I_1 \cap I_2$ and is defined by $I_1 \cap I_2 = \{x \mid x = \min(a,b)$, for $a \in I_1(i,j)$, $b \in I_2(r,c)$ and $i = r$ and $j = c\}$.

Proposition 8: Dilating an image set which is an intersection of two image sets with a structuring element set is the subset of the intersection of the dilation of the two image sets. i.e. $J_{(I_1\cap I_2),H} \subseteq J_{I_1,H} \cap J_{I_2,H}$.

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Proof : Let $x(r_0,c_0) \in \max\limits_{\substack{(r-i,c-j)\in I_1 \\ (r_1-i,c_1-j)\in I_2}} \left[\min\limits_{(i,j)\in H} \{ (I_1(r-i,c-j) \cap I_2(r_1-i,c_1-j)), H(i,j) \} \right]$

Then $x(r_0,c_0) = \max[\min(a,b)$, for some $a \in I_1(r-i,c-j) \cap I_2(r_1-i,c_1,j)$ and $b \in H(i,j)$.

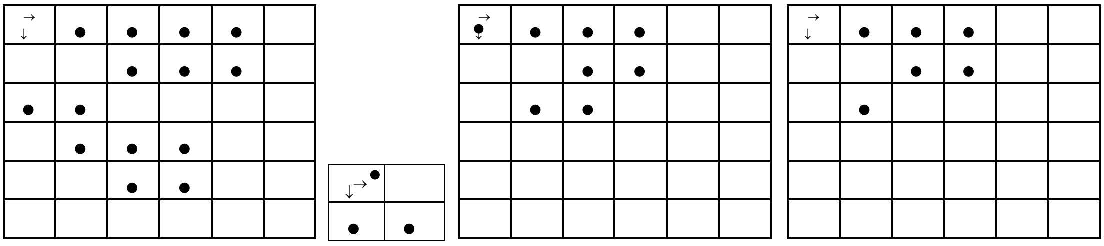
Now $a \in I_1(r-i,c-j) \cap I_2(r_1-i,c_1,j)$ means $a = \max(e,d)$ where $e \in I_1(r-i,c-j)$ and $d \in I_2(r_1-i,c_1-j)$.

Then $x(r_0,c_0) = \max[\min\{\min(e,d),b\}] = \max[\min\{\min(e,b),\min(d,b) = \min[\max\{\min(e,b), \max\{\min(d,b)\}]$.

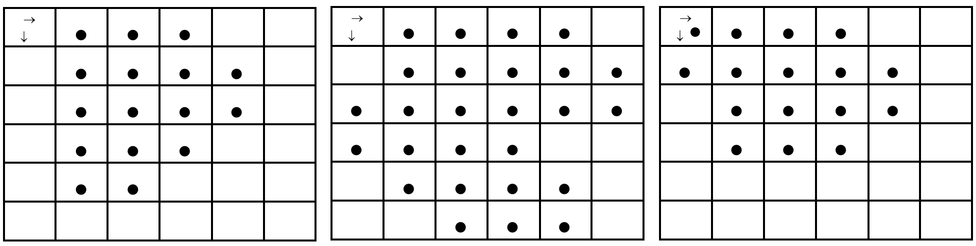
Hence $x(r_0,c_0) \in \left(\max\limits_{(r-i,c-j)\in I_1} \left[\min\limits_{(i,j)\in H} \{ I_1(r-i,c-j), H(i,j) \} \right] \right) \cap \left(\max\limits_{(r_1-i,c_1-j)\in I_2} \left[\min\limits_{(i,j)\in H} \{ I_2(r_1-i,c_1-j), H(i,j) \} \right] \right)$

Therefore $J_{(I_1 \cap I_2),H} \subseteq J_{I_1,H} \cap J_{I_2,H}$.

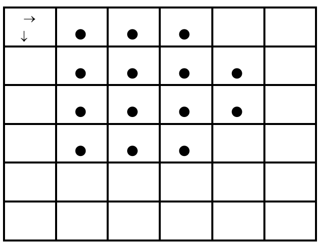
Example :



I_1 H I_2 $I_1 \cap I_2$



$J_{(I_1 \cap I_2),H}$ $J_{I_1,H}$ $J_{I_2,H}$



$J_{I_1,H} \cap J_{I_2,H}$

Proposition 9: Dilating with a structuring element set, which can be represented as an intersection of two sets is the subset of the intersection of the dilation. i.e. $J_{I,(H \cap S)} \subseteq J_{I_1,H} \cap J_{I_2,H}$.

Proof : $x(r_0,c_0) \in J_{I,(H\cap S)} \Rightarrow x(r_0,c_0) \in \max\limits_{(r-i,c-j)\in I} \left[\min\limits_{(i,j)\in H\cap S} \{ I(r-i,c-j), (H(i,j) \cap S(i,j)) \} \right]$

Then $x(r_0,c_0) = \max[\min\{a,b\}]$ for some $a \in I(r-i,c-j)$ and $b \in H(i,j) \cap S(i,j)$.

Now $b \in H(i,j) \cap S(i,j)$ means $b = \min(e,d)$ where $e \in H(i,j)$ and $d \in S(i,j)$.

Therefore $x(r_0,c_0) = \max[\min\{a, \min(e,d)\}] = \max[\min\{\min(a,e), \min(a,d)\}]$

$= \min(\max[\min\{a,e\}, \min\{a,d\}]) = \min(\max[\min\{a,e\}], \max[\min\{a,d\}])$

$\therefore x(r_0,c_0) \in \left(\max\limits_{(r-i,c-j)\in I} \left[\min\limits_{(i,j)\in H} \{ I(r-i,c-j), H(i,j) \} \right] \cap \max\limits_{(r-i,c-j)\in I} \left[\min\limits_{(i,j)\in S} \{ I(r-i,c-j), S(i,j) \} \right] \right)$

Hence $J_{I,(H\cap S)} \subseteq J_{I_1,H} \cap J_{I_2,H}$.

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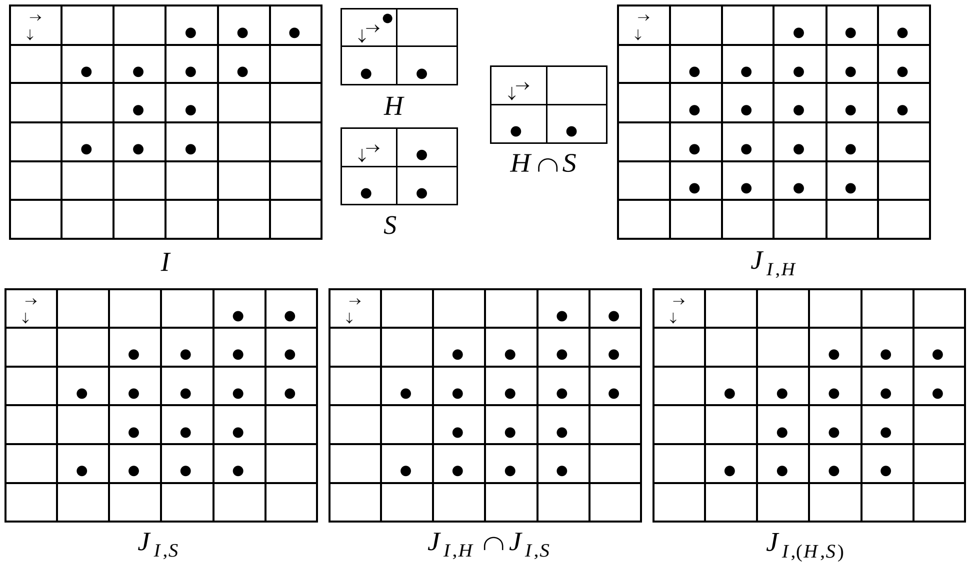
Example :



3.2 Erosion

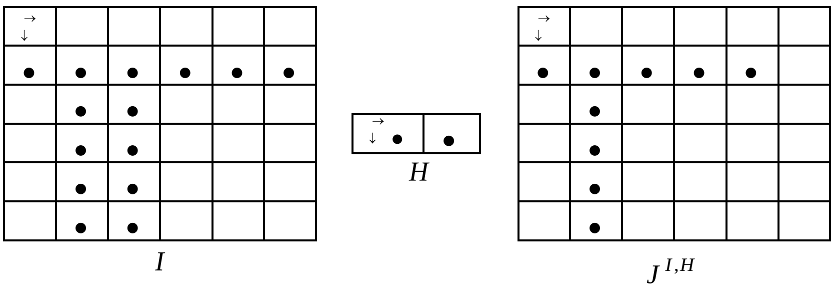
Erosion is the same as dilation except the maximum operation. The minimum operator for erosion operation replaces the maximum operator in dilation. Hence its complexity is same as dilation.

Definition 7: Let $I(r,c)$ and $H(i,j)$ be subsets of F. The erosion of $I(r,c)$ by $H(i,j)$ is denoted by $J^{I,H}(r,c)$ or $J^{I,H}$ and is defined by

$$J^{I,H} = J^{I,H}(r,c) = \min_{(r+i,c+j)\in I}\left[\min_{(i,j)\in H}\{I(r+i,c+j),H(i,j)\}\right]$$
$$= \{x \mid x = \min[\min(a,b)], \text{ for every } b \in H(i,j) \text{ and } a \in I(r+i,c+j)\}$$

where $I(r+i,c+j)$ and $H(i,j)$ are the grey values of the image element at $(r+i,c+j)$ and the structuring element at (i,j). It also converges to the definition of erosion for binary morphology. The definition is called the general definition of erosion.

Example:



This example shows the operation of erosion. Origins are placed at the arrow positions of the image set and structuring element set and the pixels with a black dot (•) have a non-zero grey value. The proposed definition of erosion satisfies the erosion of binary morphology. Figs. 3(a) and 3(b) shows the binary image set and the structuring element image set of sizes 64×64 and 9×9 respectively. The eroded image is shown in Fig.3(c). It can be easily prove the following propositions for erosion operations.

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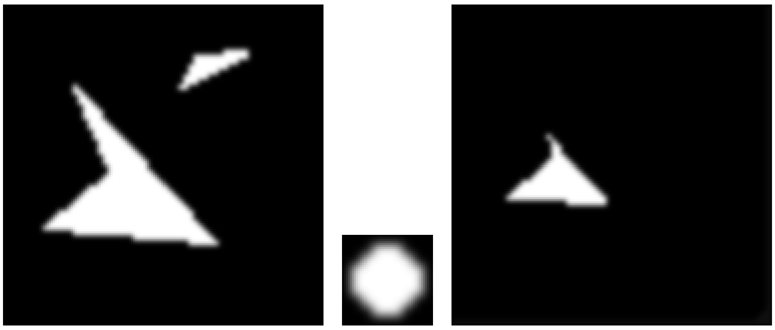

(a) (b) (c)

Fig. 3: Binary image (a) Original image of size 64×64, (b) structuring element of size 9×9, (c) eroded image

Proposition 10: Let $I(r,c)$ and $H(i,j)$ be subsets of F then $J^{I,H} = \cap_{Z \in H} I_Z$.

Proof: Let $x(r_0,c_0) \in J^{I,H}$

$$\Rightarrow x(r_0,c_0) \in \min_{(r+i,c+j)\in I}\left[\min_{(i,j)\in H}\{I(r+i,c+j),H(i,j)\}\right]$$

$\Rightarrow x(r_0,c_0) = \min[\min(a,b)]$, for every $b \in H(i,j)$ and $a \in I(r+i,c+j)$

Structuring element is always constructed with the considering pixels and the values of the considering pixels are 1 for binary image. So it is true that $a \le b$.

Now $x(r_0,c_0) = a \in I(r+i,c+j)$. Therefore, $x(r_0,c_0) \in I_Z(r,c)$ where $Z = (i,j)$.

Hence, for every $b \in H(i,j)$ and $a \in I(r+i,c+j)$, $x(r_0,c_0) \in \cap_{Z\in H} I_Z$.

So $J^{I,H} \subseteq \cap_{Z\in H} I_Z$ (3)

Similarly we can prove $\cap_{Z\in H} I_Z \subseteq J^{I,H}$ (4)

Hence, from (3) and (4) we have $J^{I,H} = \cap_{Z\in H} I_Z$.

The result of a shifted image eroded by a structuring element is same as the image eroded by the structuring element and then being shifted. This property is called translation invariance of erosion.

Proposition 11: Erosion is translation invariance i.e. $J^{I_X,H} = \left(J^{I,H}\right)_X$ and $J^{I,H_X} = \left(J^{I,H}\right)_{-X}$.

Proof: $y(r_0,c_0) \in J_{I_X,H}$ if and only if for every $b \in H(i,j)$ and $a \in I_X(r+i,c+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$. Let $X = (r_1,c_1)$. Now $a \in I_X(r+i,c+j)$ implies $a \in I(r+i+r_1,c+j+c_1)$ i.e. $a \in I(r+r_1+i,c+c_1+j)$. So $y(r_0,c_0) \in J_{I_X,H}$ if and only if for every $b \in H(i,j)$ and $a \in I(r+r_1+i,c+c_1+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$.

$$\text{i.e. } y(r_0,c_0) \in J^{I_X,H} \Leftrightarrow y(r_0,c_0) \in \min_{(r+r_1+i,c+c_1+j)\in I}\left[\min_{(i,j)\in H}\{I(r+r_1+i,c+c_1+j),H(i,j)\}\right]$$

$$\Leftrightarrow y(r_0,c_0) \in J^{I,H}(r+r_1,c+c_1)$$

$$\Leftrightarrow y(r_0,c_0) \in \left(J^{I,H}\right)_{X=(r_1,c_1)}$$

Therefore $J^{I_X,H} = \left(J^{I,H}\right)_X$.

For the second part, $y(r_0,c_0) \in J^{I,H_X}$ if and only if for every $b \in H_X(i,j)$ and $\exists$ an $a \in I(r+i,c+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$. Let $X = (r_1,c_1)$. Then $-X = (-r_1,-c_1)$. Now $b \in H_X(i,j)$ implies $b \in H(i+r_1,j+c_1)$. So $y(r_0,c_0) \in J^{I,H_X}$ if and only if for every $b \in H(i+r_1,j+c_1)$ and $\exists$ an $a \in I(r+i,c+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$.

Now replace i by $i - r_1$ and j by $j - c_1$.

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Therefore, $y(r_0,c_0) \in J^{I,H_X}$ if and only if for every $b \in H(i,j)$ and $\exists$ an $a \in I(r-r_1+i, c-c_1+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$.

$$\Leftrightarrow y(r_0,c_0) \in \min_{(r-r_1+i,c-c_1+j)\in I}\left[\min_{(i,j)\in H}\{I(r-r_1+i,c-c_1+j), H(i,j)\}\right]$$

$$\Leftrightarrow y(r_0,c_0) \in J^{I,H}(r-r_1, c-c_1)$$

$$\Leftrightarrow y(r_0,c_0) \in \left(J^{I,H}\right)_{-X}$$

Therefore $J^{I,H_X} = \left(J^{I,H}\right)_{-X}$.

If image I_1 is contained in image I_2, then the erosion of I_1 also contained in the erosion of I_2. This property is called the increasing property of erosion.

Proposition 12: Erosion is increasing i.e. If $I_1(r+i,c+j) \subseteq I_2(k+i,l+j)$ then $J^{I_1,H} \subseteq J^{I_2,H}$.

Proof : Let $y(r_0,c_0) \in J^{I_1,H}$. Then for every $b \in H(i,j)$ and $\exists$ an $a \in I_1(r+i,c+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$. Since $I_1(r+i,c+j) \subseteq I_2(k+i,l+j)$, so $a \in I_2(k+i,l+j)$. Therefore, for every $b \in H(i,j)$ and $\exists$ an $a \in I_2(k+i,l+j)$ such that $y(r_0,c_0) = \min[\min(a,b)]$. Then by definition of erosion $y(r_0,c_0) \in J^{I_2,H}$. Therefore, $J^{I_1,H} \subseteq J^{I_2,H}$.

The difference between the dilation and erosion is illuminated in the algebraic properties of erosion as contrasted with the dilation. First, the erosion of a set which itself has a decomposition as the intersection of two sets is the intersection of the erosions.

Proposition 13: Erosion of a set which itself has a decomposition as the intersection of two sets is the intersection of the erosions i.e. $J^{(I_1 \cap I_2),H} = J^{I_1,H} \cap J^{I_2,H}$.

Proof: Let $x(r_0,c_0) \in J^{(I_1\cap I_2),H}$

$$\Rightarrow x(r_0,c_0) \in \min_{\substack{(r+i,c+j)\in I_1\\(r_1+i,c_1+j)\in I_2}}\left[\min_{(i,j)\in H^P}\{(I_1(r+i,c+j)\cap I_2(r_1+i,c_1+j)), H(i,j)\}\right]$$

Then for every $b \in H(i,j)$ $\exists$ an $a \in I_1(r+i,c+j) \cap I_2(r_1+i,c_1+j)$ such that $x(r_0,c_0) = \min[\min(a,b)]$. Now $a \in I_1(r+i,c+j) \cap I_2(r_1+i,c_1+j)$ means $a = \min(e,d)$ for $a \in I_1(r+i,c+j)$ and $d \in I_2(r_1+i,c_1+j)$.

Therefore, $x(r_0,c_0) = \min[\min\{\min(e,d),b\}] = \min[\min\{\min(e,b),\min(d,b)\}]$

$= \min[\min\{\min(e,b)\},\min\{\min(d,b)\}]$

Now by definition of erosion and intersection we have

$$x(r_0,c_0) \in \left(\min_{(r+i,c+j)\in I_1{}^P}\left[\min_{(i,j)\in H^P}\{I_1(r+i,c+j), H(i,j)\}\right]\right)$$

$$\cap\left(\min_{(r+i,c+j)\in I_1{}^P}\left[\min_{(i,j)\in H^P}\{I_1(r+i,c+j), H(i,j)\}\right]\right)$$

Therefore, $x(r_0,c_0) \in J^{I_1,H} \cap J^{I_2,H}$

So $J^{(I_1 \cap I_2),H} \subseteq J^{I_1,H} \cap J^{I_2,H}$ (5)

Let $y(r_1,c_1) \in J^{I_1,H} \cap J^{I_2,H}$. Now by the definition of intersection $y(r_1,c_1) = \min(a,b)$ for $a \in J^{I_1,H}$ and $b \in J^{I_2,H}$. From the definition of erosion $a \in J^{I_1,H}$ means for every $d \in H(i,j)$ $\exists$ an $e \in I_1(r+i,c+j)$ such that $a = \min[\min(e,d)]$.

Similarly, $b \in J^{I_2,H}$ means for every $d \in H(i,j)$ $\exists$ a $g \in I_2(r_1+i,c_1+j)$ such that $b = \min[\min(g,d)]$.

Therefore, $y(r_1,c_1) = \min[\min\{\min(e,d)\},\min\{\min(g,d)\}] = \min[\min\{\min(e,d),\min(g,d)\}]$

$= \min[\min\{\min(e,g),d\}]$

Now by definition of erosion and intersection

$$y(r_1,c_1) \in J^{(I_1 \cap I_2),H}$$

Therefore, $J^{I_1,H} \cap J^{I_2,H} \subseteq J^{(I_1 \cap I_2),H}$ (6)

Therefore by equations (5) and (6) we have

$$J^{(I_1 \cap I_2),H} = J^{I_1,H} \cap J^{I_2,H}$$

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On the other hand, whereas the dilation of the union of two images is equal to the union of their dilation (Proposition 6), for the erosion this relationship is one of containment.

Proposition 14: Erosion of the union of two images is subset to the union of their erosion i.e. $J^{I_1,H} \cup J^{I_2,H} \subseteq J^{(I_1 \cup I_2),H}$.

Proof : Let $x(r_0,c_0) \in J^{I_1,H} \cup J^{I_2,H}$. Then for $a \in J^{I_1,H}$ and $b \in J^{I_2,H}$ such that x = max(a,b). Now $a \in J^{I_1,H}$ means for every $d \in H(i,j)$ and $\exists$ an $a \in I_1(r+i,c+j)$ such that a = min[min(e,d)]. Similarly, $b \in J^{I_2,H}$ means for every $d \in H(i,j)$ and $\exists$ a $g \in I_2(r_1+i,c_1+j)$ such that b = min[min(g,d)].

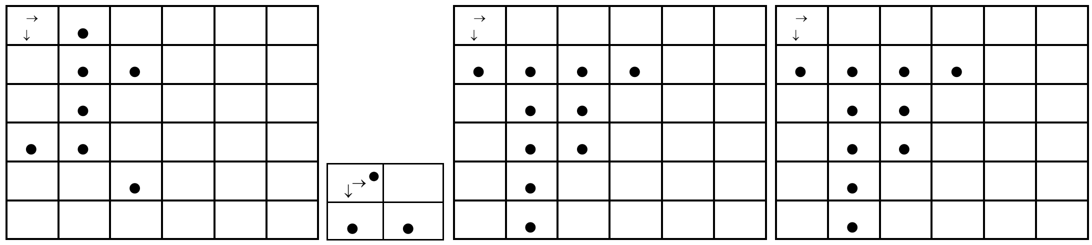
Therefore, $x(r_0,c_0) =$ max[min{min(e,d)},min{min(g,d)}] = max[min{min(e,g),d}]

= min[min{max(e,g),d}]

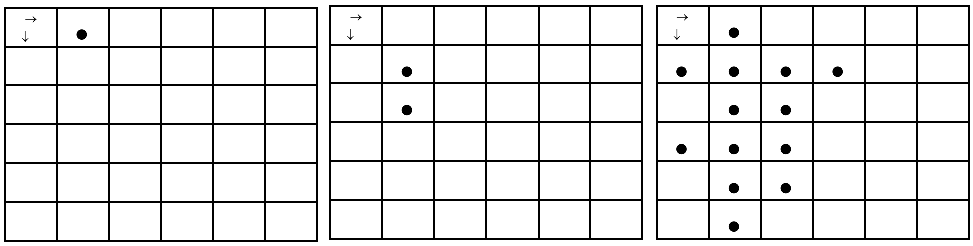
Then by definition of union and erosion $x(r_0,c_0) \in J^{(I_1 \cup I_2),H}$

Therefore $J^{I_1,H} \cup J^{I_2,H} \subseteq J^{(I_1 \cup I_2),H}$

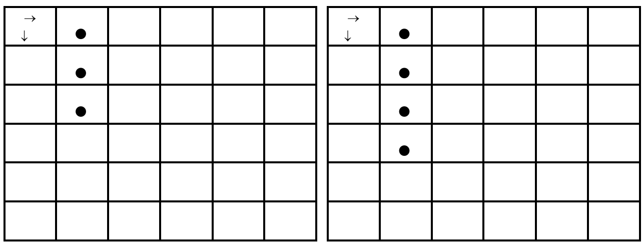
Example :



I_1 H I_2 I_2



$J^{I_1,H}$ $J^{I_2,H}$ $I_1 \cup I_2$



$J^{I_1,H} \cup J^{I_2,H}$ $J^{(I_1 \cup I_2),H}$

There is a relationship between the order of dilation and erosion operations. Suppose there are two structuring element sets and one image set. First result is obtained by eroded the two structuring element sets and then dilated the image set with the result of the eroded image. Second result is obtained by dilated the image set and any one of the two structuring element sets and then eroded the dilated result with another structuring element set. It is very interesting that the first operation result is a subset of second operation result. In some sense this indicates that when performing erosion and dilation, performing erosion first is more severe than performing dilation first.

Proposition 15: $I_1 \subseteq J^{I_2,H} \Leftrightarrow J_{I_1,H} \subseteq I_2$.

Proof: Suppose $I_1 \subseteq J^{I_2,H}$. Let $x(r_0,c_0) \in J_{I_1,H}$. Then for some $a \in I_1(r-i,c-j)$ and $b \in H(i,j)$ such that $x(r_0,c_0)$ = max[min(a,b)]. Now since $I_1 \subseteq J^{I_2,H}$ so $a \in J^{I_2,H}$. Then for every $b \in H(i,j)$) $\exists$ an $e \in I_2(r+i,c+j)$ such that a = min[min(e,b)]. Therefore, $x(r_0,c_0)$ = max[min{min[min(e,b)],b}]. Now for particular e < b, $x(r_0,c_0) = e \in I_2$. Therefore $J_{I_1,H} \subseteq I_2$. Again suppose $J_{I_1,H} \subseteq I_2$.

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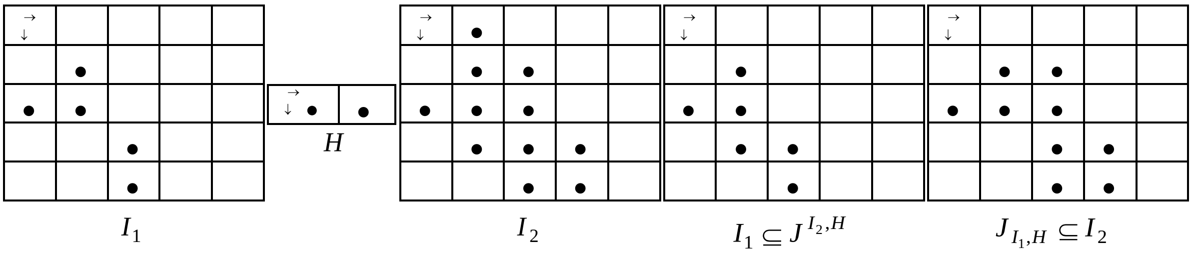
Let $y(r_0,c_0)\in I_1$ and $b\in H(i,j)$ such that $\max[\min\{y(r_0,c_0),b\}]\in J_{I_1,H}$. But $J_{I_1,H}\subseteq I_2$. Therefore, $\max[\min\{y(r_0,c_0),b\}]\in I_2$.

Now for every $b\in H(i,j)\exists$ an element $\max[\min\{y(r_0,c_0),b\}]\in I_2$ such that $\min[\min\{\max[\min\{y(r_0,c_0),b\}],b\}]\in J^{I_2,H}$. Now for particular $y(r_0,c_0)<b$, $\min[\min\{\max[\min\{y(r_0,c_0),b\}],b\}] = y(r_0,c_0)\in J^{I_2,H}$. Therefore, $I_1\subseteq J^{I_2,H}$.

Example:



There is a relationship between the order of dilation and erosion operations. Suppose there are two structuring element sets and one image set. First result is obtained by eroded the two structuring element sets and then dilated the image set with the result of the eroded image. Second result is obtained by dilated the image set and any one of the two structuring element sets and then eroded the dilated result with another structuring element set. It is very interesting that the first operation result is a subset of second operation result. In some sense this indicates that when performing erosion and dilation, performing erosion first is more severe than performing dilation first.

Proposition 16: $J_{I,}{}^{(H,S)}\subseteq J_{(I,H),}{}^{S}$

Proof: Let $x(r_0,c_0)\in J_{I,}^{(H,S)}$. Then for some $a\in I(r-i,c-j)$ and $b\in J^{H,S}$ such that $x(r_0,c_0)$ = max[min(a,b)]. Now $b\in J^{H,S}$ means for every $d\in S(k,l)$ and $\exists$ an $e\in H(i+k,j+l)$ such that b = min[min(e,d)].

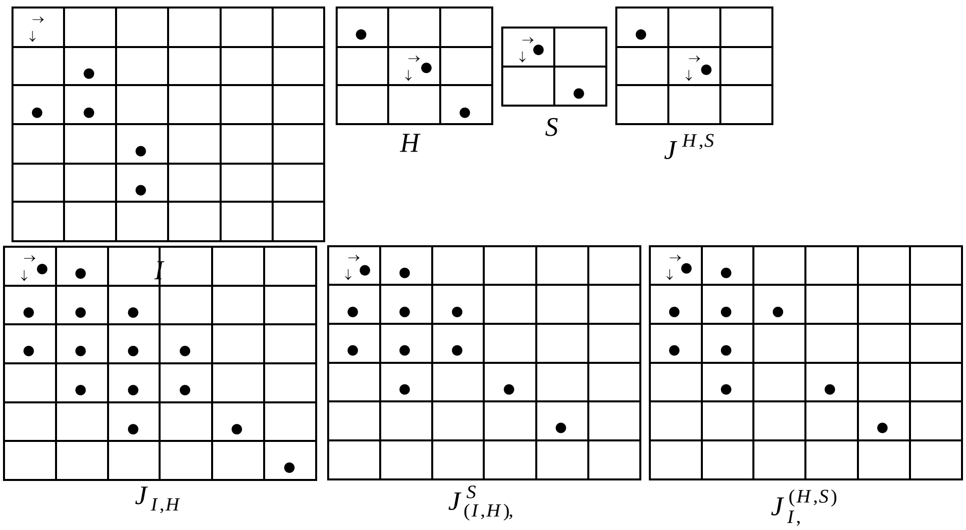
Therefore, $x(r_0,c_0)$ = max[min{a, min[min(e,d)]}] for some $a\in I(r-i,c-j)$, $e\in H(i+k,j+l)$ and every $d\in S(k,l)$.

Now replacing i by i-k and j by j-l. So $x(r_0,c_0)$ = min[min{max[min(a,e)],d}] for some $a\in I(r+k-i,c+l-j)$, $e\in H(i,j)$ e∈H(i,j) and every $d\in S(k,l)$.

Therefore, $x(r_0,c_0)\in J^{S}_{(I,H),}$.

Hence, $J_{I,}{}^{(H,S)}\subseteq J_{(I,H),}{}^{S}$

Example:



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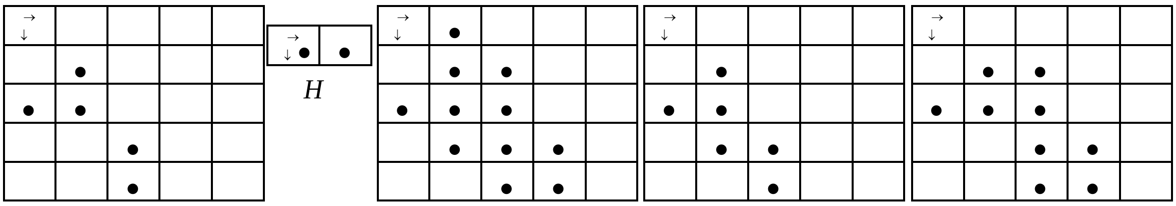
Proposition 17: $I_1 \subseteq J^{I_2,H} \Leftrightarrow J_{I_1,H} \subseteq I_2$

Proof: Suppose $I_1 \subseteq J^{I_2,H}$. Let $x(r_0,c_0) \in J_{I_1,H}$. Then for some $a \in I_1(r-i,c-j)$ and $b \in H(i,j)$ such that $x(r_0,c_0) =$ max[min(a,b)]. Now since $I_1 \subseteq J^{I_2,H}$ so $a \in J^{I_2,H}$. Then for every $b \in H(i,j) \exists$ an $e \in I_2(r+i,c+j)$ such that a = min[min(e,b)]. Therefore, $x(r_0,c_0)$ = max[min{min[min(e,b)],b}]. Now for particular $e < b$, $x(r_0,c_0) = e \in I_2$. Therefore $J_{I_1,H} \subseteq I_2$.

Again suppose $J_{I_1,H} \subseteq I_2$. Let $y(r_0,c_0) \in I_1$ and $b \in H(i,j)$ such that max[min{ $y(r_0,c_0), b$ }]$\in J_{I_1,H}$. But $J_{I_1,H} \subseteq I_2$. Therefore, max[min{ $y(r_0,c_0), b$ }]$\in I_2$

Now for every $b \in H(i,j) \exists$ an element max[min{ $y(r_0,c_0), b$ }]$\in I_2$ such that min[min{ max[min{ $y(r_0,c_0), b$ }],b}]$\in J^{I_2,H}$. Now for particular $y(r_0,c_0) < b$, min[min{max[min { $y(r_0,c_0), b$ }],b}] = $y(r_0,c_0) \in J^{I_2,H}$. Therefore, $I_1 \subseteq J^{I_2,H}$.

Example:



I_1 H I_2 $J^{I_2,H}(I_1 \subseteq J^{I_2,H})$ $J_{I_1,H}(J_{I_1,H} \subseteq I_2)$

4. Opening and Closing

Now we are ready to understand another reason why dilation and erosion have an essential connection to shape. Dilation and erosion are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or an erosion of the image followed by the dilation of the eroded result. In either case, the result of successively applied dilation and erosion is an elimination of specific image detail smaller than the structuring element without the global geometric of unsuppressed features.

It is very interesting fact that image transformations employing successively applied dilation and erosion are idempotent. That is, their repeated application effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of the structuring elements under which they can opened or can be closed and yet remain the same (Haralick *et al*., 1987).

Definition 8: The *opening* of image $I(r,c)$ by the structuring element $H(i,j)$ is denoted by $(I)_H$ and is defined by

$$(I)_H = \max_{(r-i,c-j)\in I}\left[\min_{(i,j)\in H}\left\{\left(\min_{(r+k,c+l)\in I}\left[\min_{(k,l)\in H}\{I(r-i+k,c-j+l),H(k,l)\}\right]\right),H(i,j)\right\}\right]$$

Definition 9: The *closing* of image $I(r,c)$ by the structuring element $H(i,j)$ is denoted by $(I)^H$ and is defined by

$$(I)^H = \min_{(r+k,c+l)\in I}\left[\min_{(k,l)\in H}\left\{\left(\max_{(r-i,c-j)\in I}\left[\min_{(i,j)\in H}\{I(r+k-i,c+l-j),H(i,j)\}\right]\right),H(k,l)\right\}\right]$$

Note that if $I(r,c)$ is unchanged by opening it with $H(i,j)$, we say that $I(r,c)$ is open with respect to $H(i,j)$ and if $I(r,c)$ is unchanged by closing it with $H(i,j)$ we say that $I(r,c)$ is closed with respect to $H(i,j)$. The opening and closing of the image Fig. 1(a) by the structuring element Fig. 1(b) are shown in Fig. 4(a) and Fig. 4(b) respectively.

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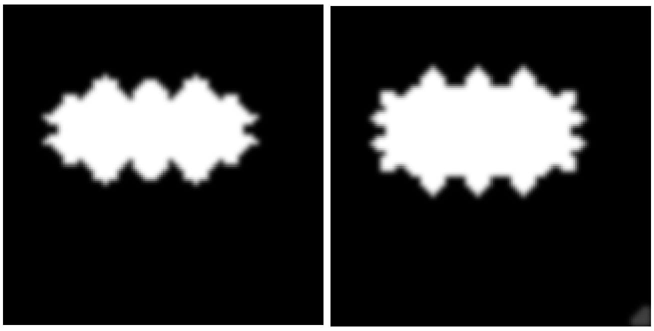
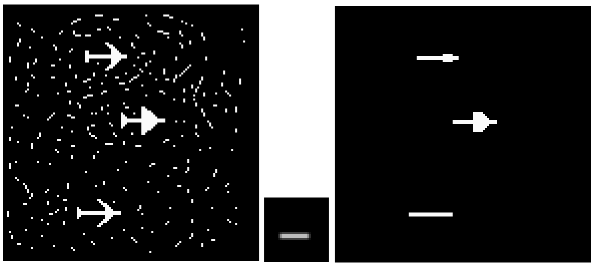


(a) (b)

Fig. 4: (a) Image of size 64×64 after opening operation, (b) Image of size 64×64 after closing operation

Fig. 5(a) shows an image of size 128×128. The image is a noisy image with some aeroplane like objects. Fig. 5(b) is the structuring element of size 5×1. The opening of the image Fig. 5(a) by the structuring element Fig. 5(b) is shown in Fig.5(c). The objects are properly extracted and the noise is removed.



(a) (b) (c)

Fig. 5: A noisy binary image (a) Noisy image of size 128×128, (b) Structuring element of size 5×1, (c) Image after opening operation

Image sets dilated by the structuring element set remain invariant under an opening with the same structuring element set and also invariant under dilation followed by the closing by the same structuring element set.

Proposition 18: $J_{I,H} = \left(J_{I,H}\right)_H = J_{(I)^H,H}$

Proof: Let $M(r,c) = J_{I,H} = \max\limits_{(r-i,c-j)\in I}\left[\min\limits_{(i,j)\in H}\{I(r-i,c-j), H(i,j)\}\right]$,

$$N(r_1,c_1) = J^{M,H} = \min\limits_{(r_1+k,c_1+l)\in M}\left[\min\limits_{(k,l)\in H}\{M(r_1+k,c_1+l), H(k,l)\}\right] \text{ and}$$

$$Q(r_{11},c_{11}) = J_{N,H} = \max\limits_{(r_{11}-m,c_{11}-n)\in N}\left[\min\limits_{(m,n)\in H}\{N(r_{11}-m,c_{11}-n), H(m,n)\}\right]$$

Now by Proposition 16, $M(r,c) = J_{I,H}$ implies $I(r_1,c_1) \subseteq J^{M,H} = N(r_1,c_1)$.

Now $N(r_1,c_1) = J^{M,H}$ implies

$$M(r,c) \supseteq J_{N,H} = Q(r_{11},c_{11}) \tag{7}$$

But $I(r_1,c_1) \subseteq N(r_1,c_1)$ implies $J_{I,H} \subseteq J_{N,H}$

$$\text{i.e. } M(r,c) \subseteq Q(r_{11},c_{11}) \tag{8}$$

From (7) and (8) we have $M(r,c) = Q(r_{11},c_{11})$

$$\text{i.e. } \max\limits_{(r-i,c-j)\in I}\left[\min\limits_{(i,j)\in H}\{I(r-i,c-j), H(i,j)\}\right] = \max\limits_{(r_{11}-m,c_{11}-n)\in N}\left[\min\limits_{(m,n)\in H}\{N(r_{11}-m,c_{11}-n), H(m,n)\}\right] \tag{9}$$

Now the right hand side of (9)

$$= \max\limits_{(r_{11}-m,c_{11}-n)\in M}[\min\limits_{(m,n)\in H}\{(\min\limits_{(r_{11}+k,c_{11}+l)\in M}[\min\limits_{(k,l)\in H}\{M(r_{11}-m+k, c_{11}-n+l), H(k,l)\}]), H(m,n)\}]$$

$$= \max\limits_{(r_{11}-m,c_{11}-n)\in I}[\min\limits_{(m,n)\in H}\{(\min\limits_{(r_{11}+k,c_{11}+l)\in I}[\min\limits_{(k,l)\in H}\{(\max\limits_{(r_{11}-i,c_{11}-j)\in I}[\min\limits_{(i,j)\in H}\{I(r_{11}-m+k-i, c_{11}-n+l-j), H(i,j)\}]), H(k,l)\}]), H(m,n)\}] \tag{10}$$

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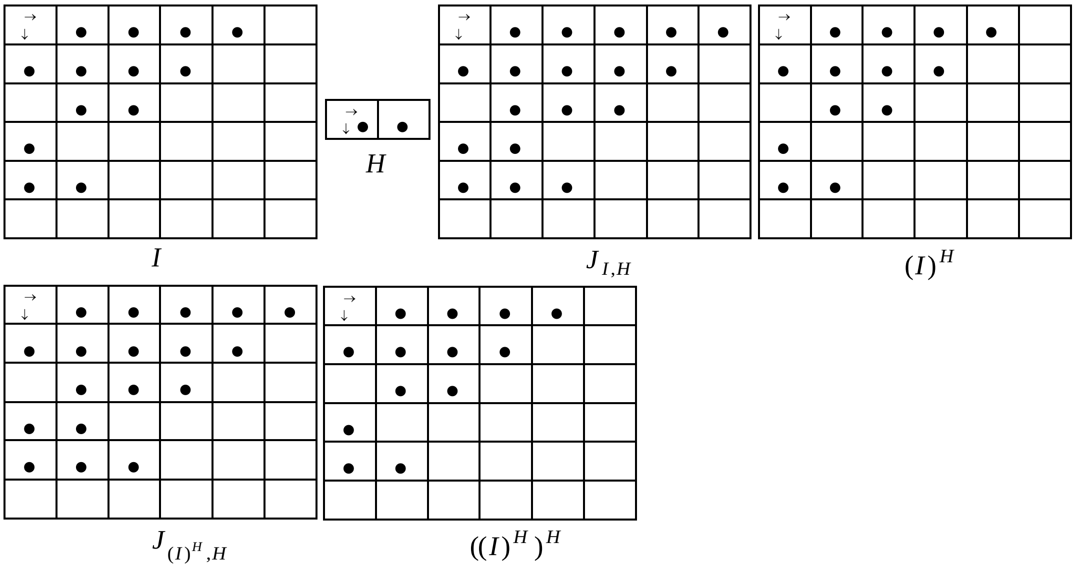
Replacing r_{11} by r and c_{11} by c we have $J_{I,H} = (J_{I,H})_H$.
Similarly the last result can be proved from equation (10) by interchanging the dummy index (i, j) and (m,n).
Hence $J_{I,H} = \left(J_{I,H}\right)_H = J_{(I)^H,H}$.

Proposition 19: Idempotency of closing i.e. $\left((I)^H\right)^H = (I)^H$.
Proof: It is known by Proposition 18 that $J_{I,H} = J_{(I)^H,H}$. Let $J_{I,H} = M$ and $J_{(I)^H,H} = N$.
Now eroded both sides by the structuring element $H(i, j)$, we have $J^{M,H} = J^{N,H}$. Therefore, $\left((I)^H\right)^H = (I)^H$.

Example:



Similarly, image sets eroded by the structuring element set remain invariant under a closing with the same structuring element set and also invariant under erosion followed by the opening by the same structuring element set.

Proposition 20: $J^{I,H} = J^{(I)_H,H} = \left(J^{I,H}\right)^H$

Proof: Let $M(r,c) = \min\limits_{(r+i,c+j)\in I}\left[\min\limits_{(i,j)\in H}\{I(r+i,c+j), H(i,j)\}\right] = J^{I,H}$,

$$N(r_1,c_1) = \max_{(r_1-k,c_1-l)\in M}\left[\min_{(k,l)\in H}\{M(r_1-k,c_1-l), H(k,l)\}\right] = J_{M,H} \text{ and}$$

$$Q(r_{11},c_{11}) = \min_{(r_{11}+m,c_{11}+n)\in N}\left[\min_{(m,n)\in H}\{N(r_{11}+m,c_{11}+n), H(m,n)\}\right] = J^{N,H}$$

Now by Proposition 17, $M(r,c) = J^{I,H} \Rightarrow I(r_1,c_1) \supseteq J_{M,H} = N(r_1,c_1)$

$N(r_1,c_1) = J_{M,H} \Rightarrow M(r,c) \subseteq J^{N,H} = Q(r_{11},c_{11})$ (11)

Again $I(r_1,c_1) \supseteq N(r_1,c_1)$ implies $J^{I,H} \supseteq J^{N,H}$

Therefore, $M(r,c) \supseteq Q(r_{11},c_{11})$ (12)

From (11) and (12) we have $M(r,c) = Q(r_{11},c_{11})$

i.e. $$\min_{(r+i,c+j)\in I}\left[\min_{(i,j)\in H}\{I(r+i,c+j), H(i,j)\}\right] = \min_{(r_{11}+m,c_{11}+n)\in N}\left[\min_{(m,n)\in H}\{N(r_{11}+m,c_{11}+n), H(m,n)\}\right] \quad (13)$$

Now The right hand side of (13)

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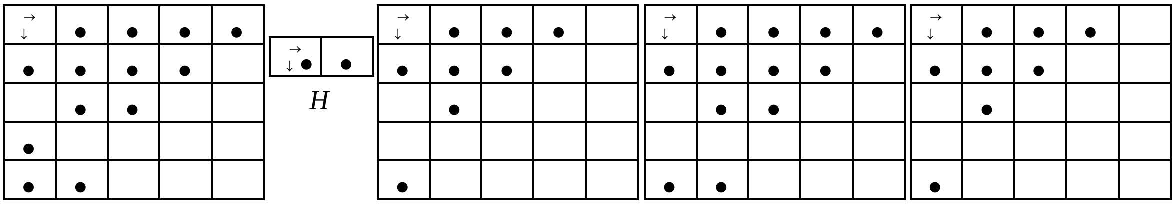
$$= \min_{(r_{11}+m,c_{11}+n)\in M}[\min_{(m,n)\in H}\{(\max_{(r_{11}-k,c_{11}-l)\in M}[\min_{(k,l)\in H}\{M(r_{11}+m-k,c_{11}+n-l),H(k,l)\}]),H(m,n)\}]$$

$$= \min_{(r_{11}+m,c_{11}+n)\in I}[\min_{(m,n)\in H}\{(\max_{(r_{11}-k,c_{11}-j)\in I}[\min_{(k,l)\in H}\{(\min_{(r_{11}+i,c_{11}+j)\in I}[\min_{(i,j)\in H}\{I(r_{11}+m-k+i,\mathrm{c}_{11}+n-l+j),$$
$$H(i,j)\}]),H(k,l)\}]),H(m,n)\}]$$

Replacing r_{11} by r and c_{11} by c, we have $J^{I,H}=\left(J^{I,H}\right)^H$.
The other result will be obtained by interchanging (m,n) and (i,j).
Hence $J^{I,H}=J^{(I)_H,H}=\left(J^{I,H}\right)^H$

Example:



I $\quad H \quad$ $J^{I,H}$ $\quad (I)_H \quad$ $J^{(I)_H,H}=(J^{I,H})^H=J^{I,H}$

Proposition 21: Idempotency of opening i.e. $(I)_H=((I)_H)_H$.

Proof: We know by Proposition 20, $J^{I,H}=J^{(I)_H,H}$. Let $J^{I,H}=M$ and $J^{(I)_H,H}=N$.
Now dilate both sides by the structuring element $H(i,j)$, we have $J_{M,H}=J_{N,H}$.

Therefore, $(I)_H=((I)_H)_H$

The closing is increasing. That is, if I_1 and I_2 are two image sets and I_2 is contained in I_1, then the closing of the image set I_2 by the structuring element set H will necessarily be contained in the closing of the image set I_1 by the same structuring element set H.

Proposition 22: Closing is increasing i.e. if $I_1(r,c)\subseteq I_2(r_1,c_1)$ then $(I_1)^H\subseteq(I_2)^H$.

Proof: Let $I_1(r,c)\subseteq I_2(r_1,c_1)$. Let $(x_0,y_0)\in(I_1)^H$.

Then $x(r_0,c_0)\in \min_{(r+k,c+l)\in I_1}[\min_{(k,l)\in H}\{(\max_{(r-i,c-j)\in I_1}[\min_{(i,j)\in H}\{I_1(r+k-i,c+l-j),H(i,j)\}]),H(k,l)\}]$

Then for every $b\in H(k,l)$ and $\exists$ an $a\in \max_{(r-i,c-j)\in I_1}\left[\min_{(i,j)\in H}\{I_1(r+k-i,c+l-j),H(i,j)\}\right]$ such that $x(r_0c_0)=\min[\min(a,b)]$. But

$a\in \max_{(r-i,c-j)\in I_1}\left[\min_{(i,j)\in H}\{I_1(r+k-i,c+l-j),H(i,j)\}\right]$ means for some $e\in I_1(r+k-i,c+l-j)$ and $d\in H(i,j)$ such that $a=\max[\min(e,d)]$.

Therefore, $x(r_0,c_0)=\min[\min\{\max[\min(e,d)],b\}]$.

Now since $I_1(r,c)\subseteq I_2(r_1,c_1)$ and $e\in I_1(r+k-i,c+l-j)$, therefore $e\in I_2(r+k-i,c+l-j)$.

So for some $e\in I_2(r+k-i,c+l-j)$ and $d\in H(i,j)$ such that $a=\max[\min(e,d)]$.

Therefore, $a\in \max_{(r-i,c-j)\in I_2}\left[\min_{(i,j)\in H}\{I_2(r+k-i,c+l-j),H(i,j)\}\right]$

Nor for every $b\in H(k,l)$ and $\exists$ an $a\in \max_{(r-i,c-j)\in I_2}\left[\min_{(i,j)\in H}\{I_2(r+k-i,c+l-j),H(i,j)\}\right]$ such that $x(r_0c_0)=\min[\min(a,b)]$.

Therefore, $(x_0,y_0)\in(I_2)^H$.

Hence $(I_1)^H\subseteq(I_2)^H$.

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Similarly the opening is increasing. That is, if I_1 and I_2 are two image sets and I_2 is contained in I_1, then the opening of the image set I_2 by the structuring element set H will necessarily be contained in the opening of the image set I_1 by the same structuring element set H.

Proposition 23: Opening is increasing i.e. if $I_1(r,c) \subseteq I_2(r_1,c_1)$ then $(I_1)_H \subseteq (I_2)_H$.
Proof: Let $I_1(r,c) \subseteq I_2(r_1,c_1)$. Let $(x_0, y_0) \in (I_1)_H$.
Then $x(r_0,c_0) \in \max_{(r-i,c-j)\in I_1} [\min_{(i,j)\in H} \{(\min_{(r+k,c+l)\in I_1} [\min_{(k,l)\in H} \{I_1(r-i+k, c-j+l), H(k,l)\}]), H(i,j)\}]$

Therefore, for some $a \in \min_{(r-i+k,c-j+l)\in I_1}\left[\min_{(k,l)\in H} \{I_1(r+k-i, c+l-j), H(k,l)\}\right]$ and $b \in H(i,j)$ such that $x(r_0 c_0) = \min[\min(a,b)]$.

Now $a \in \min_{(r-i+k,c-j+l)\in I_1}\left[\min_{(k,l)\in H} \{I_1(r+k-i, c+l-j), H(k,l)\}\right]$ means for every $d \in H(k,l)$ and $\exists$ an $e \in I_1(r-i+k, c-j+l)$ such that $a = \min[\min(e,d)]$.
Therefore, $x(r_0,c_0) = \max[\min\{\min[\min(e,d)], b\}]$.
Now since $I_1(r,c) \subseteq I_2(r_1,c_1)$ and $e \in I_1 r-i+k, c-j+l)$, so $e \in I_2 r-i+k, c-j+l)$.
Now for every $d \in H(k,l)$ and $\exists$ an $e \in I_2 r-i+k, c-j+l)$ such that $a = \min[\min(e,d)]$.

Therefore, $a \in \min_{(r-i+k,c-j+l)\in I_2}\left[\min_{(k,l)\in H} \{I_2(r-i+k, c-j+l), H(k,l)\}\right]$.

Now $a \in \min_{(r-i+k,c-j+l)\in I_2}\left[\min_{(k,l)\in H} \{I_2(r-i+k, c-j+l), H(k,l)\}\right]$ and $b \in H(i,j)$ such that $x(r_0,c_0) = \max[\min(a,b)]$.
Therefore, $x(r_0,c_0) \in (I_2)_H$.
Hence $(I_1)_H \subseteq (I_2)_H$.

5. Conclusion

Mathematical morphology is useful to the processing of digital images that are based on shape. The new definitions of dilation and erosion for binary image are proposed in this paper. The opening, closing and their relations are also proposed in this paper. We have shown that morphological dilation and erosion satisfy most of the set theory operations. We have also shown that morphological opening and closing are increasing and idempotent. Various experimental results are shown the efficiency of the definitions.

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