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SOME COMMON FIXED POINT THEOREMS AND RESULTS IN FUZZY METRIC SPACES

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Abstract

This paper presents some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces. We also introduce the concept of R-weak commutativity of type in fuzzy metric spaces. Some related results and illustrative examples are also discussed.

Keywords: Occasionally weakly compatible mappings, fuzzy metric space, fixed point, R-weakly commuting mappings.

1 Introduction

Fuzzy set was defined by Zadeh [27]. Kramosil and Michalek [15] introduced fuzzy metric space, George and Veermani [7] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki [26] proved fixed point theorems for R-weakly commutating mappings. It proved a turningpoint in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [20] which laid the foundation of fuzzy mathematics. Consequently the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities for fixed point theorists.

They also showed that every metric induces a fuzzy metric and by now there exists considerable literature on this topic.

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2 Preliminaries

Definition 2.1: A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2: A binary operation $\succ$: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if {[0, 1], $\succ$} is an abelian topological monoid with unit 1 such that $a \succ b \leq c \succ d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 2.3: A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is a continuous t-norms if * is satisfying conditions:
(i) $*$ is an commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 2.4: A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$, (**f**1) $M(x, y, t) > 0$;
(**f**2) $M(x, y, t) = 1$ if and only if $x = y$
(**f**3) $M(x, y, t) = M(y, x, t)$;
(**f**4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(**f**5) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.5 (Induced fuzzy metric): Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.6: Let $(X, M, *)$ be a fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in X is said to converges to x in X if for each $\_ > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \_$ for all $n \geq n_0$.
(b) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\_ > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \_$ for all $n, m \geq n_0$.
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.7: A pair of self-mappings (**f**, g) of a fuzzy metric space $(X, M, *)$ is said to be

(i) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all $x \in X$ and $t > 0$.
(ii) R-weakly commuting if there exists some $R > 0$ such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

Definition 2.8: Two self maps f and g of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Lemma 2.9: Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 2.10: Let X be a set, f, g selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.11: Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 2.12: A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, >)$ is said to be

(i) weakly commuting (cf.[18]) if $M(fgx, gfx, t) \geq M(fx, gx, t)$,

(ii) R-weakly commuting (cf.[18]) if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R),$$

(iii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that $M(fgx, ggx, t) \geq M(fx, gx, t/R)$,

(iv) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that $M(gfx, ffx, t) \geq M(fx, gx, t/R)$,

(v) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that $M(ffx, ggx, t) \geq M(fx, gx, t/R)$, for all $x \in X$ and $t > 0$.

Lemma 2.13: Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

3 Main Results

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \quad (1)$$

for all $x, y \in X$ and for all $t > 0$, then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (1)

$$\begin{aligned} M(Ax, By, qt) &\geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\} \\ &= \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ &\quad M(Ax, By, t), M(By, Ax, t)\} \\ &= M(Ax, By, t). \end{aligned}$$

Therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$$\begin{aligned} M(w, z, qt) &= M(Aw, Bz, qt) \\ &\geq \min\{M(Sw, Tz, t), M(Sw, Az, t), M(Bz, Tz, t), \\ &\quad M(Aw, Tz, t), M(Bz, Sw, t)\} \\ &= \min\{M(w, z, t), M(w, z, t), M(z, z, t), \\ &\quad M(w, z, t), M(z, w, t)\} \\ &= M(w, z, t) \end{aligned}$$

Therefore we have $z = w$ by Lemma 2.14 and z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (1).

Theorem 3.2: let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \geq \varphi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}) \quad (2)$$

for all $x, y \in X$ and $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S and T.

Proof: The proof follows from Theorem 3.1.

Theorem 3.3: let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \geq \varphi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)) \quad (3)$$

for all $x, y \in X$ and $\varphi : [0, 1]^5 \to [0, 1]$ such that $\varphi(t, 1, 1, t, t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc, there are points point $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. By inequality (3) we have

$$\begin{aligned} M(Ax, By, qt) &\geq \varphi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)) \\ &= \varphi(M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), M(Ax, By, t), M(By, Ax, t)) \\ &= \varphi(M(Ax, By, t), 1, 1, M(Ax, By, t), M(By, Ax, t)) \\ &> M(Ax, By, t). \end{aligned}$$

a contradiction, therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Tx$ is the unique point of coincidence of A and T. By Lemma 2.13 w is a unique common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Thus z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (3).

Theorem 3.4: let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc. If there exists a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$

$$M(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t), \quad (4)$$

then there exists a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. By inequality (4) we have

$$\begin{aligned} M(Ax, By, qt) &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(Ax, Ty, t) \\ &= M(Ax, By, t) * M(Ax, Ax, t) * M(By, By, t) \\ &\quad * M(Ax, By, t) \\ &\geq M(Ax, By, t) * 1 * 1 * M(Ax, By, t) \\ &\geq M(Ax, By, t) \end{aligned}$$

Thus we have $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (4) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Thus w is a common fixed point of A, B, S and T.

Corollary 3.5: let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc. If there exists a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$

$$\begin{aligned} M(Ax, By, qt) &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(By, Sx, 2t) * M(Ax, Ty, t), \qquad (5) \end{aligned}$$

then there exists a unique common fixed point of A, B, S and T.

Proof: We have

$$\begin{aligned} M(Ax, By, qt) &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(By, Sx, 2t) * M(Ax, Ty, t) \\ &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * \\ &\quad M(Sx, Ty, t) * M(Ty, By, t) * M(Ax, Ty, t) \\ &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(Ax, Ty, t) \end{aligned}$$

and therefore from Theorem 3.4, A, B, S and T have a common fixed point.

Theorem 3.6: let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied

(i) $AX \subset TX \cap SX$
(ii) the pairs $\{A, S\}$ and $\{A, T\}$ are weakly compatible,
(iii) there exists a point $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$\begin{aligned} M(Ax, Ay, qt) &\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Ay, Ty, t) \\ &\quad * M(Ax, Ty, t) \qquad (6) \end{aligned}$$

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Then A, S and T have a unique common fixed point.

Proof: Since compatible implies owc, the result follows from 3.4

Theorem 3.7: let $(X, M, *)$ be a complete fuzzy metric space and let A and B be self-mappings of X. Let the A and B are owc. If there exists a point $q \in (0,1)$ for all $x, y \in X$ and $t > 0$

$$M(Sx, Sy, qt) \geq \alpha M(Ax, Ay, t) + \beta \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\} \quad (7)$$

for all $x, y \in X$, where $\alpha, \beta, > 0, \alpha + \beta > 1$. Then A and S have a unique common fixed point.

Proof: Let the pairs $\{A, S\}$ be owc, so there is a point $x \in X$ such that $Ax = Sx$. Suppose that there exist another point $y \in X$ for which $Ay = Sy$. We claim that $Sx = Sy$. By inequality (8) we have

$$\begin{aligned} M(Sx, Sy, qt) &\geq \alpha M(Ax, Ay, t) + \beta \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\} \\ &= \alpha M(Sx, Sy, t) + \beta \min\{M(Sx, Sy, t), M(Sx, Sx, t), M(Sy, Sy, t)\} \\ &= (\alpha + \beta) M(Sx, Sy, t) \end{aligned}$$

a contradiction, since $(\alpha + \beta) > 1$. Therefore $Sx = Sy$. Therefore $Ax = Ay$ and Ax is unique. From Lemma 2.14, A and S have a unique fixed point.

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